Controllability of quantum walks on graphs

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In recent years, quantum walks on graphs have emerged as one of the most useful protocols to design quantum algorithms. The study of the controllability of these systems refer to the analysis of the set of states the system can be in. In this talk, I will consider discrete time quantum walks on graphs with coin. These systems consist of two coupled quantum systems: a walker system whose states are in correspondence with the nodes of the underlying graph and a coin system whose states indicate the directions for the motion on the graph. The evolution consists of two operations at each step: a coin tossing operation which is a unitary operation on the space of the coin only and a conditional shift operation which changes the state of the walker system according to the current value of the coin. The talk deals with the case where the coin operation can be changed at every time step and presents a study of the set of possible evolutions and states. The first result discussed says that the set of available unitary transformations is a Lie group whose Lie algebra can be described explicitly in every case. The system is called completely controllable if the set of available evolutions is the full unitary group. I give both Lie algebraic and combinatorial tests to check complete controllability. In particular, the combinatorial tests are based on the construction of an auxiliary graph whose connectedness is equivalent to the controllability of the given system. This test avoids Lie algebraic calculations which typically involve commutators of very large matrices. I prove that controllability only depends on the underlying graph and not on the specific quantum walk considered on it. It is in particular always verified for complete graphs and product graphs of controllable systems. In view of this dynamical and control theoretic analysis, in the second part of the talk, I take a different look at two issues of current interest in quantum information: quantum algorithms and approximation of continuous dynamics by discrete time quantum walks. A quantum algorithm is a sequence of unitary operations to transfer the state of a quantum system from an initial value to a desired final value. As such, it can be seen as a control algorithm. I provide general constructive algorithms to transfer between two arbitrary states for a quantum walk. These consist of appropriate sequences of coin tossing operations and conditional shifts. I give an upper bound on the number of steps needed for an arbitrary transfer which depends on the features of the underlying graph. Furthermore, I discuss the interplay of these results with two types of algorithms of current interest in quantum information: search algorithms on a graph and algorithms to generate certain outputs with a prescribed probability distribution which are at the heart of classical randomized algorithms. As for the approximation of continuous time dynamics by discrete time quantum walks, the controllability analysis leads to the study of a special Lie algebra of skew-Hermitian operators on the full space for the coin and walker systems. The continuous time evolutions corresponding to Hamiltonians in this Lie algebra (modulo multiplication by the imaginary unit) can be reproduced using the discrete time quantum walk. This can be achieved both exactly and approximately with various constructive methods. I illustrate this using an example: a quantum walk on a 2-dimensional periodic lattice. Of particular interest are dynamics that correspond to continuous time quantum walks on the same graph. The described results give a general method to obtain the continuous time quantum walk as a limit of the discrete time quantum walk and, in that, answer an open question in quantum information theory. Moreover they offer tools to compare the performance of the discrete and continuous quantum walks in several cases. [In collaboration with Dr. F. Albertini at the University of Padova, Italy]